90. Proposed by J. MASCUS BOORMAN, Consultative Mechanician and Counselor at Law, Woodmere, Long Island, N. Y.

Solve $\quad x^2+y^2+m(x+y)=m^2 \ldots\ldots (1).$

$$x^2+y^2+xy=m^2 \ldots\ldots (2).$$

I. Solution by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.; A. H. BELL, Hillsboro, Ill.; HON. JOSIAH H. DRUMMOND, LL. D., Portland, Me.; CHARLES C. CROSS, Libertytown, Md.; ELMER SCHUYLER, High Bridge, N. J.; and M. A. GRUBER, A. M., Washington, D. C.

Subtracting we have $m(x+y)=xy \ldots\ldots (3)$.

Squaring (3) $m^2(x^2+2xy+y^2)=x^2y^2$.

Multiplying (2) by m^2, $m^2(x^2+xy+y^2)=m^4$.

Whence $x^2y^2-m^2xy=m^4 \ldots\ldots (4)$,

Solving as a quadratic, $xy=\frac{m^2}{2}(1\pm\sqrt{5}) \ldots\ldots (5)$.

Adding (5) and (2), $x^2+2xy+y^2=\frac{m^2}{4}(6\pm2\sqrt{5})$.

Whence $x+y=\pm\frac{m}{2}(\sqrt{5}\pm1)$.

Similarly, $x-y=\pm\frac{m}{2}(\sqrt{-2\mp6\sqrt{5}})$.

Whence $x=\pm\frac{m}{4}(\sqrt{5}\pm1+\sqrt{-2\mp6\sqrt{5}})$,

$$y=\pm\frac{m}{4}(\sqrt{5}\pm1-\sqrt{-2\mp6\sqrt{5}}),$$

II. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

(1)$-$(2) gives $m(x+y)=xy$. Whence $x+y=xy/m$.

Put $x=vy$. Then $vy+y=vy^2/m$, and $v+1=vy/m$.

Whence $y=m(v+1)/v$, and $x=vy=m(v+1)$.

Substituting the values of x and y in (1) or (2), we have

$$m^2(v+1)^2+\frac{m^2(v+1)^2}{v^2}+\frac{m^2(v+1)^2}{v}=m^2.$$

Dividing by m^2, clearing of fractions, expanding, etc., we obtain

$$v^4+3v^3+4v^2+3v+1=v^2.$$

Adding and subtracting v^3+2v^2+v, we find

$$(v+1)^4-v(v+1)^2=v^2.$$

Completing square and extracting square root, we get

$$2(v+1)^2-v=\pm v\sqrt{5}.$$

Expanding, transposing, and uniting terms, we obtain

$$2v^2+(3\mp\sqrt{5})v=-2.$$

Completing square, extracting square root, etc., we find

$$v=\tfrac{1}{4}[-3\pm\sqrt{5}\pm\sqrt{-2(1\pm3\sqrt{5})}].$$

Whence $v+1=\tfrac{1}{4}[1\pm\sqrt{5}\pm\sqrt{-2(1\pm3\sqrt{5})}]$,

and $(v+1)/v=\tfrac{1}{4}[1\pm\sqrt{5}\mp\sqrt{-2(1\pm3\sqrt{5})}]$.

$\therefore\ x=m(v+1)$ and $y=m[(v+1)/v]$.

III. Solution by H. C. WHITAKER, A. M., Ph. D., Professor of Mathematics, Manual Training School, Philadelphia, Pa., and ALOIS F. KOVARIK, Instructor in Mathematics and Physics, Decorah Institute, Decorah, Iowa.

Let $x=v+z$, and $y=v-z$. $2v^2+2z^2+2mv=m^2$. $3v^2+z^2=m^2$.

Whence $v=\tfrac{1}{4}(1\pm\sqrt{5})m$, and $z=\pm\tfrac{1}{4}\sqrt{(-2\mp6\sqrt{5})}m$.

The four values of x and y are

$x=\tfrac{1}{4}[1+\sqrt{5}+\sqrt{(-2-6\sqrt{5})}]m$; $y=\tfrac{1}{4}[1+\sqrt{5}-\sqrt{(-2-6\sqrt{5})}]m$.

$x=\tfrac{1}{4}[1+\sqrt{5}-\sqrt{(-2-6\sqrt{5})}]m$; $y=\tfrac{1}{4}[1+\sqrt{5}+\sqrt{(-2-6\sqrt{5})}]m$.

$x=\tfrac{1}{4}[1-\sqrt{5}+\sqrt{(-2+6\sqrt{5})}]m$; $y=\tfrac{1}{4}[1-\sqrt{5}-\sqrt{(-2+6\sqrt{5})}]m$.

$x=\tfrac{1}{4}[1-\sqrt{5}-\sqrt{(-2+6\sqrt{5})}]m$; $y=\tfrac{1}{4}[1-\sqrt{5}+\sqrt{(-2+6\sqrt{5})}]m$.

IV. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa., and J. SCHEFFER, A. M., Hagerstown, Md.

Let $x+y=t$, $xy=r$.

Then (1) and (2) become $t^2-2r+mt=m^2$......(3), $t^2-r=m^2$......(4).

Subtracting (4) from (3) we get $r=mt$......(5), or $t=r/m$......(6).

(5) in (3) gives $t^2-mt=m^2$. $\therefore\ t=\tfrac{1}{2}m(1\pm\sqrt{5})$.

(6) in (4) gives $r^2-m^2r=m^4$. $\therefore\ r=\tfrac{1}{2}m^2(1\pm\sqrt{5})$.

Since $x+y=t$ and $xy=r$, we have

$x=\tfrac{1}{2}[t\pm\sqrt{(t^2-4r)}]=\tfrac{1}{4}m\{1\pm\sqrt{5}\pm\sqrt{[-2(1\pm3\sqrt{5})]}\}$.

$y=\tfrac{1}{2}[t\mp\sqrt{(t^2-4r)}]=\tfrac{1}{4}m\{1\pm\sqrt{5}\mp\sqrt{[-2(1\pm3\sqrt{5})]}\}$.

V. Solution by the PROPOSER.

x/m, y/m ; with *eight* roots (four each) [or sixteen *by* $\sqrt{m^2}=\pm m$] ; $x:y=$ ratio $X:Y$ with values $x=X$ times *any-* or *every-*thing ; $y=Y$ *any-* or *every-*thing ; $\therefore\ x/X=y/Y=m$ of an infinite degree and roots, *all* values from $\pm(1/\infty^n)$ to 0 to $\pm\infty^\infty$. And that m is a *blind* factor in $x=Xm$; $y=Ym$;—SOLVE CIRODE's curious PROBLEM. (I)$-$(II)......$m(x+y)=xy$......(III). *Un*-factor $x=Xm$; $y=Ym$......(A). $\therefore\ X^2m^2+Y^2m^2+m(Xm+Ym)=m^2$......($I_1$) ; $m(Xm+Ym)=XmYm$......(III_1) *divided* by m^2 are $X^2+Y^2+x+y=1$..... (I_a) ;

$X+Y=XY\ldots\ldots(\mathrm{III}_a)$. *Twice* $(\mathrm{III}_a)+(\mathrm{I}_a)$ gives $(X+Y)^2-(X+Y)=1\ldots\ldots$ (IV). $\therefore X+Y=\frac{1}{2}(1\pm\sqrt{5})\ldots\ldots$(V). $\therefore$ By (III_a) $XY=\frac{1}{2}(1\pm5)\ldots\ldots$(VI) ! $\therefore$ (V)(VI), (by *nature*) the constants of $X^2-\frac{1}{2}(1\pm\sqrt{5})X+\frac{1}{2}(1\pm\sqrt{5})=0\ldots\ldots$ (VII) whose *four* roots are by *its origin* (by *turns*) the *eight* roots of *both* X and Y. Multiplying (VII+)(VII−) ; is $X^4-X^3+2X=0\ldots\ldots$(B) *two* real roots

each$\ldots\ldots\left\{\begin{array}{l} X=\frac{1}{4}[1-\sqrt{5}+\sqrt{(-2+6\sqrt{5})}]=Y_1 \\ X_1=\frac{1}{4}[1-\sqrt{5}+\sqrt{(-2+6\sqrt{5})}]=Y \end{array}\right\}\ldots\ldots$(VIII). *Two* imag-

inery each $\ldots\ldots\left\{\begin{array}{l} X_2=\frac{1}{4}[1-\sqrt{5}+\sqrt{(-1-6\sqrt{5})}]=Y_a \\ X_a=\frac{1}{4}[1+\sqrt{5}-\sqrt{(-2-6\sqrt{5})}]=Y_2 \end{array}\right\}=0{,}809016{,}994375$

$\pm3.926373{,}373101{,}2946\sqrt{-1}\ldots\ldots$(IX).

$$\therefore\left\{\begin{array}{l} X=-1.153721{,}375541{,}766 \text{ to } Y=0.535687\ldots\ldots; \\ X_1=0.535687{,}386791{,}872 \text{ to } Y_1=-1.153721\ldots\ldots; \end{array}\right.$$

$$\left.\begin{array}{l} X_2=.809016{,}99\ldots\ldots+3.926373,\ldots\ldots\sqrt{-1}=Y_a \\ X_a=.809016{,}99\ldots\ldots-3.926373,\ldots\ldots\sqrt{-1}=Y_2 \end{array}\right\}\ldots\ldots(\mathrm{C}).$$

$\therefore X+Y=X_1+Y_1=-0.618033{,}988750-;\ \ X_2+Y_2=1.618033{,}988750-=X_a+$ $Y_a\ldots\ldots$(D). *Proof* by $XY=X+Y\ldots\ldots(\mathrm{III}_a)\left\{\begin{array}{l} X=-1.15372137\ldots\ldots \\ Y=0.53568738\ldots\ldots \end{array}\right\}$

$$\left.\begin{array}{l} \text{Product}=-0.6180339{,}88+=XY \\ \text{Sum}\quad=-0.6180339{,}89-=X+Y \end{array}\right\}\ldots\ldots(\mathrm{XI}).\ \text{Also (C) and (A).}$$

$$\left\{\begin{array}{l} x=Ym=m(-1.1537\ldots\ldots \\ y=Ym=m(\ \ 0.5356\ldots\ldots \end{array}\right\}\begin{array}{l} \text{Product, } mXmY=xy=in\ x \\ \text{Sum,}\quad mX+mY=x+y \end{array}$$

$$\left.\begin{array}{r} y=m_+(x+y)=m^2(-0.61803\ldots\ldots) \\ \therefore \text{Multiplying}=m=m(x+y)=m^2(-0.61802\ \ldots\ldots) \end{array}\right\}\ldots\ldots(\mathrm{XII}),$$

*because (III)$\ldots\ldots$ $xy=m(x+y)=m^2(-0.618033.98\ldots\ldots)=m^2XY\ldots\ldots$(E). $\therefore$ m (*vanishes*) is only a *blind* factor in x and y. $\therefore$ by (E) ratio $x:y=X:Y$ and and (I) (II) are *fully* solved, process, ratio and roots. Q. E. D. [As, solve (III) for m. $\therefore m=\frac{1}{2}(-1\mp\sqrt{5})(x+y)\ldots\ldots$ *i. e.* $1=\frac{1}{2}(-1\mp\sqrt{5})(X+Y)\ldots\ldots\therefore m$ and x and y vanish together as *always*].

COROLLARY. Two quadratics in *three* unknowns, *if* their every term be quadrate (in unknowns) stand for, *and solve*, the two bi-quadratics each in one *true* unknown found by treating two stated unknowns as if multiples (or other functions) of the third, and un-factoring them from it (into two *new* letters). This treatment also determines the third unknown, roots and equation, or their nature and forms when inexpressible.

COROLLARY 2. *Strictly* $\sqrt{m}=\pm m$. $\therefore x^2+y^2\pm(x+y)\ldots\ldots(\mathrm{I}_c)$ with (II_a) have *eight* roots each, viz.: (VIII, IX) or (C) direct, and *also* with signs reversed.

SCHOLIUM. *General* form $x^2+y^2+c(x+y)=a\ldots\ldots$(A) ; $x^2+y^2+xy=b$ $\ldots\ldots(B)$ solves (by quadratics) many biquadrates, decomposes surd compounds (other than of 1 with $\sqrt{5}$), finds roots, etc.